

## Distribution of the maximum of second order concomitants of selected order statistics from the Morgenstern family of distributions

Biju Thomas <sup>a</sup> Preethi John <sup>b</sup> and Liz John <sup>c</sup>

<sup>a</sup>Department of Statistics, Sree Sankara College, Kalady, Kerala, India-683574 <sup>b</sup>Department of Statistics, Vimala College, Thrissur, Kerala, India -680 009

<sup>c</sup>Department of Statistics, Nirmala College, Muvattupuzha, Kerala, India 686 661

### ARTICLE HISTORY

Compiled December 4, 2025

Received 03 June 2025; Accepted 25 October 2025

### ABSTRACT

This paper extends the theory of second order concomitants proposed by Scaria and Thomas (2013). It investigates the small and large sample theoretical properties of a statistic derived from these concomitants, specified by

$$V_{(\frac{m+1}{2}, s:m); l, k} = \max \left\{ Y_{(\frac{m+1}{2}, s:m); [k-l+1:k]}, \dots, Y_{(\frac{m+1}{2}, s:m); [k:k]} \right\}, \quad l = 1, 2, \dots, k$$

based on the Morgenstern family of bivariate distributions. The results are applied to a selection problem and the estimation of the scale parameter of a marginal variable in the bivariate uniform distribution.

### KEYWORDS

Order statistics, second order concomitants, Morgenstern family, bivariate uniform, selection procedure, estimation

## 1. Introduction

Let  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  denote a random sample of size  $n$  drawn from a continuous bivariate population  $(X, Y)$  characterized by the cumulative distribution function (c.d.f.)  $F_{X,Y}(x, y)$ . When the sample is ordered according to the  $X$ -component, the order statistics  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$  are obtained. The  $Y$ -values corresponding to the order statistics  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$  are denoted by  $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ , and are referred to as the concomitants of order statistics. David (1973) introduced the notion of concomitants, which was independently formulated by Bhattacharya (1974) under the term induced order statistics. Extensive research has explored the distributional behavior of concomitants, assuming the random variables are i.i.d. Key contributions include works by David (1973), David and Galambos (1974) and Yang (1977). Other notable studies include Bairamov et al. (2001), Scaria and Nair (2005, 2008), Beg and Ahsanullah (2008), Scaria and Thomas (2008), Ahsanullah (2009), Scaria and Mohan

(2021) and Thomas and Scaria (2011). Further, Eryilmaz (2005) investigated the behaviour of concomitants in the context of independent but non-identically distributed random vectors. Comprehensive reviews of the literature on concomitants can be found in David and Nagaraja (1998) and more recently in Koshti and Kamalja (2024).

Scaria and Thomas (2013) introduced second order concomitants of order statistics to minimize measurements of highly expensive or difficult-to-measure objects. The concept pertains to two correlated variables,  $X$  and  $Y$ , where  $X$  is easily observable, while  $Y$  requires substantial effort or expense to measure. Suppose we have a bivariate sample of size  $n = mk$ , where  $k$  is a positive integer. Randomly partition this sample into  $k$  subsamples, each comprising  $m$  observations. Let  $Y_{[r:m]i}$  and  $Y_{[s:m]i}$  denote the concomitants associated with the order statistics  $X_{r:m}$  and  $X_{s:m}$  respectively in the  $i^{\text{th}}$  subsample, where  $(i = 1, 2, \dots, k)$ . The joint distribution of  $(Y_{[r:m]i}, Y_{[s:m]i})$  is the same as that of  $(Y_{[r:m]}, Y_{[s:m]})$  for all  $i = 1, 2, \dots, k$ . Obtain the measurements  $Y_{[r:m]i}$  from each of the  $k$  subsamples, where  $i = 1, 2, \dots, k$ . Let the observations  $Y_{[r:m]}$  be arranged in ascending order. Then  $Y_{[r:m];p:k}$  denotes the  $p^{\text{th}}$  order statistic among the concomitants  $Y_{[r:m]}$ . The observation on the  $Y_{[s:m]}$  variate that is paired with  $Y_{[r:m];p:k}$  is the concomitant of  $Y_{[r:m];p:k}$ . This value is defined as the  $p^{\text{th}}$  second order concomitant of  $X_{r:m}$ , represented by  $Y_{(r,s:m);[p:k]}$ . Evidently  $Y_{(r,s:m);[p:k]}$  is the  $p^{\text{th}}$  first order concomitant of  $Y_{[r:m];p:k}$  and  $p^{\text{th}}$  second order concomitant of  $X_{r:m}$ . An important benefit of employing second order concomitants in sampling is the significant reduction in the number of measurements required for variables that are costly or challenging to observe. For further details, see Thomas (2015). The technique of second order concomitants has been extensively investigated by Barakat et al. (2025), who examined their behaviour within the Cambanis family and explored associated information measures.

Morgenstern (1956) constructed a family of bivariate distributions  $F_{X,Y}(x, y)$  specified by the marginal distributions  $F_X(x)$  and  $F_Y(y)$  and association parameter  $\alpha$  of the form

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))], \quad -1 \leq \alpha \leq 1. \quad (1)$$

The Morgenstern family offers a general framework for modelling bivariate distributions by allowing various marginal distributions to be incorporated. When the marginal distributions are already known, utilizing the Morgenstern family can provide valuable insights into the underlying phenomenon, making it advantageous for bivariate data analysis. Since the joint distribution function  $F_{X,Y}(x, y)$  of the Morgenstern family is expressed in terms of its marginal distributions, random sampling from this family is straightforward. Consequently, these distributions are particularly suitable for simulation studies, especially in scenarios where modelling weak dependence between variables is desired. Scaria and Nair (1999) explored the distributional properties of concomitants associated with order statistics derived from the Morgenstern family of distributions. Further, Scaria and Nair (2008) examined the distribution theory concerning the maximum and minimum of the  $r^{\text{th}}$  concomitants drawn from  $k$  independent subgroups, each comprising  $m$  observations, within the same family.

The principal application of concomitants occurs in selection procedures, as outlined by Yeo and David (1984), wherein  $k$  individuals are selected from a total of  $n$  based on their  $X$ -values, while the associated  $Y$ -values constitute the primary focus of analysis. For example,  $X$  may represent the score achieved in an initial screening test, while  $Y$  denotes the score obtained following subsequent training and a second

evaluation. Accordingly,

$$V_{k,n} = \max(Y_{[n-k+1:n]}, \dots, Y_{[n:n]}),$$

for  $k = 1, 2, \dots, n$ , denotes the highest score among the  $k$  selected individuals in the second test. The ratio

$$e_{k,n} = \frac{\mathbb{E}[V_{k,n}]}{\mathbb{E}[Y_{n:n}]}$$

which evidently increases to 1 with  $k$ , serves as a measure of the effectiveness of the screening procedure. It may be desirable to select  $k$  to make this ratio sufficiently close to 1. Feinberg (1991) and Feinberg and Huber (1996) examined various properties of  $V_{k,n}$  within the context of cutoff-based selection rules under conditions of imperfect information. Feinberg (1991) employed simulation techniques to investigate the behavior of  $\mathbb{E}[V_{k,n}]$  for specific values of  $n$ , under the assumption that the sample is drawn from a bivariate normal distribution. Scaria and Nair (2005) explored the distributional behavior of  $V_{k,n}$  within the framework of the Morgenstern family, addressing both finite-sample characteristics and asymptotic properties. A comprehensive discussion is provided in Scaria (2003).

Nagaraja and David (1994) demonstrated that

$$\begin{aligned} F_{k,n}(y) &= P(V_{k,n} \leq y) \\ &= P(Y_{[n-k+1:n]} \leq y, \dots, Y_{[n:n]} \leq y) \\ &= \int_{-\infty}^{\infty} [F_{Y|X}^*(y|x)]^k f_{X_{n-k:n}}(x) dx \end{aligned} \quad (2)$$

where

$$F_{Y|X}^*(y|x) = P(Y \leq y | X > x)$$

and  $f_{X_{n-k:n}}(x)$  is the p.d.f of  $X_{n-k:n}$ .

This paper extends the concept of the maximum of concomitants associated with selected order statistics to the maximum of second order concomitants. The motivation behind this extension is to minimize costly or time-consuming measurements of the primary variable. By utilizing second order concomitants, such measurements can be significantly reduced, offering a more efficient approach.

Let

$$(Y_{[r:m]i}, Y_{[s:m]i}), \quad i = 1, 2, \dots, k$$

be the measurements on the concomitants of order statistics  $X_{r:m}$  and  $X_{s:m}$  from the  $k$  subsamples of size  $m$ . We define

$$V_{(r,s;m);l,k} = \max(Y_{(r,s;m);[k-l+1:k]}, \dots, Y_{(r,s;m);[k:k]}), \quad l = 1, 2, \dots, k \quad (3)$$

and

$$e_{(r,s:m);l,k} = \frac{\mathbb{E} [V_{(r,s:m);l,k}]}{\mathbb{E} [Y_{(r,s:m);k:k}]} \quad (4)$$

The quantity  $e_{(r,s:m);l,k}$ , which clearly increases to 1 with  $l$ , measures the effectiveness of the selection procedure using second order concomitants of order statistics. We wish to choose  $l$  such that  $e_{(r,s:m);l,k}$  is sufficiently close to 1.

Furthermore, in many cases, measuring median concomitants is cost-effective, especially when compared to focusing on the concomitants of the largest order statistics. For example, in medical studies, researchers often focus on median survival times, median treatment responses, or median values for biomarkers. Identification and measurement of extreme cases—for instance, exceptional responders or outliers with unusual biomarker levels—can be challenging, expensive, and require additional resources. Similarly, in environmental monitoring, median concentrations of pollutants are often easier to measure and record. In contrast, extreme values might require specialized equipments or more frequent sampling. In such situations, choose  $r = \frac{m+1}{2}$  in equations (3) and (4). Hence,

$$V_{(\frac{m+1}{2}, s:m);l,k} = \max \left( Y_{(\frac{m+1}{2}, s:m);[k-l+1:k]}, \dots, Y_{(\frac{m+1}{2}, s:m);[k:k]} \right), \quad (5)$$

$$l = 1, 2, \dots, k$$

and

$$e_{(\frac{m+1}{2}, s:m);l,k} = \frac{\mathbb{E} [V_{(\frac{m+1}{2}, s:m);l,k}]}{\mathbb{E} [Y_{(\frac{m+1}{2}, s:m);k:k}]} \quad (6)$$

We would like to choose  $l$  such that  $e_{(\frac{m+1}{2}, s:m);l,k}$  is close to 1.

In Section 2, we examine both the finite-sample and asymptotic distributional properties of  $V_{(\frac{m+1}{2}, s:m);l,k}$ . In Section 3, the derived results are first applied to a selection problem and subsequently extended to address the estimation of the scale parameter associated with a costly marginal variable.

## 2. Distribution of $V_{(\frac{m+1}{2}, s:m); l, k}$

From Scaria and Nair (1999), for the Morgenstern family specified by (1), the joint distribution function of  $Y_{[r:m]}$  and  $Y_{[s:m]}$ , where  $s > r$  is given by

$$\begin{aligned}
 F_{[r,s:m]}(y_1, y_2) = F_Y(y_1) F_Y(y_2) & \left\{ 1 + \alpha \left( \frac{m - 2r + 1}{m + 1} \right) [1 - F_Y(y_1)] \right. \\
 & + \alpha \left( \frac{m - 2s + 1}{m + 1} \right) [1 - F_Y(y_2)] \\
 & + \alpha^2 \left( \frac{m - 2s + 1}{m + 1} - \frac{2r(m - 2s)}{(m + 1)(m + 2)} \right) \\
 & \left. [1 - F_Y(y_1)] [1 - F_Y(y_2)] \right\}. \tag{7}
 \end{aligned}$$

The marginal distributions of  $Y_{[r:m]}$  and  $Y_{[s:m]}$  are

$$G_{Y_1}(y_1) = F_Y(y_1) \left\{ 1 + \alpha \left( \frac{m - 2r + 1}{m + 1} \right) [1 - F_Y(y_1)] \right\} \tag{8}$$

and

$$G_{Y_2}(y_2) = F_Y(y_2) \left\{ 1 + \alpha \left( \frac{m - 2s + 1}{m + 1} \right) [1 - F_Y(y_2)] \right\} \tag{9}$$

respectively.

Let

$$\begin{aligned}
 \alpha_1 &= \alpha \left( \frac{m - 2r + 1}{m + 1} \right), \\
 \alpha_2 &= \alpha \left( \frac{m - 2s + 1}{m + 1} \right), \\
 \alpha_3 &= \alpha^2 \left( \frac{m - 2s + 1}{m + 1} - \frac{2r(m - 2s)}{(m + 1)(m + 2)} \right). \tag{10}
 \end{aligned}$$

If  $r = \frac{m+1}{2}$ , then

$$\begin{aligned}
 \alpha_1 &= 0, \\
 \alpha_2 &= \alpha \left( \frac{m - 2s + 1}{m + 1} \right), \\
 \alpha_3 &= \alpha^2 \left( \frac{m - 2s + 1}{m + 1} - \frac{m - 2s}{m + 2} \right). \tag{11}
 \end{aligned}$$

In this case,

$$G_{Y_1}(y_1) = F_Y(y_1). \tag{12}$$

Hence,

$$F_{[\frac{m+1}{2}, s:m]}(y_1, y_2) = F_Y(y_1) F_Y(y_2) \left\{ 1 + \alpha_2 [1 - F_Y(y_2)] + \alpha_3 [1 - F_Y(y_1)] [1 - F_Y(y_2)] \right\}. \quad (13)$$

(13) is a subclass of the bivariate Cambanis family of distributions. Hence, the conditions for the parameters  $\alpha_2$  and  $\alpha_3$  are:

$$1 + \alpha_2 + \alpha_3 \geq 0, \quad 1 - \alpha_2 - \alpha_3 \geq 0, \quad 1 + \alpha_2 - \alpha_3 \geq 0, \quad \text{and} \quad 1 - \alpha_2 + \alpha_3 \geq 0.$$

The conditional distribution of  $Y_{[s:m]}$  given  $Y_{[\frac{m+1}{2}:m]} = y_1$  is given by

$$F_{Y_2|Y_1}(y_2 | y_1) = F_Y(y_2) [1 + \alpha_2 \{1 - F_Y(y_2)\} + \alpha_3 \{1 - 2F_Y(y_1)\} \{1 - F_Y(y_2)\}].$$

The cumulative distribution function (c.d.f.) of  $V_{([\frac{m+1}{2}, s:m]; l, k)}$  follows from (2). It is

$$\begin{aligned} F_{([\frac{m+1}{2}, s:m]; l, k)}(y_2) &= P \left[ V_{([\frac{m+1}{2}, s:m]; l, k)} \leq y_2 \right] \\ &= \int_{-\infty}^{\infty} \left[ F_{Y_2|Y_1}^*(y_2 | y_1) \right]^l f_{k-l:k}(y_1) dy_1 \end{aligned} \quad (14)$$

where

$$f_{k-l:k}(y_1) = \frac{1}{B(k-l, l+1)} [F_Y(y_1)]^{k-l-1} [1 - F_Y(y_1)]^l f_Y(y_1) \quad (15)$$

and

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

is the complete beta function. By taking  $k$  subsamples of size  $m$  each from the Morgenstern family so that  $mk = n$  and considering the concomitants  $Y_{[\frac{m+1}{2}:m]}$  and  $Y_{[s:m]}$

so that  $(Y_{[\frac{m+1}{2}:m]_i}, Y_{[s:m]_i})$  for  $i=1,2,\dots,k$  is a random sample from (13), we have

$$\begin{aligned} F_{Y_2|Y_1}^*(y_2 | y_1) &= \frac{1}{\int_{y_1}^{\infty} f_Y(y_1) dy_1} \left[ \int_{y_1}^{\infty} F_{Y_2|Y_1}(y_2 | y_1) f_Y(y_1) dy_1 \right] \\ &= \frac{1}{1 - F_Y(y_1)} \left[ \int_{y_1}^{\infty} F_Y(y_2) \left\{ 1 + \alpha_2(1 - F_Y(y_2)) \right. \right. \\ &\quad \left. \left. + \alpha_3(1 - 2F_Y(y_1))(1 - F_Y(y_2)) \right\} f_Y(y_1) dy_1 \right] \\ &= F_Y(y_2) [1 + \alpha_2(1 - F_Y(y_2)) - \alpha_3(1 - F_Y(y_2)) F_Y(y_1)]. \end{aligned} \quad (16)$$

Hence, from (14) we obtain

$$\begin{aligned} F_{(\frac{m+1}{2}, s:m); l, k}(y_2) &= \int_{-\infty}^{\infty} [F_Y(y_2) (1 + \alpha_2(1 - F_Y(y_2)) - \alpha_3(1 - F_Y(y_2)) F_Y(y_1))]^l f_{k-l:k}(y_1) dy_1 \\ &= \int_{-\infty}^{\infty} [F_Y(y_2) (1 + \alpha_2(1 - F_Y(y_2)) - \alpha_3(1 - F_Y(y_2)) F_Y(y_1))]^l \\ &\quad \times \frac{1}{B(k-l, l+1)} [F_Y(y_1)]^{k-l-1} [1 - F_Y(y_1)]^l f_Y(y_1) dy_1 \\ &= \frac{[F_Y(y_2)]^l}{B(k-l, l+1)} \int_{-\infty}^{\infty} [1 + (1 - F_Y(y_2)) (\alpha_2 - \alpha_3 F_Y(y_1))]^l \\ &\quad \times [F_Y(y_1)]^{k-l-1} [1 - F_Y(y_1)]^l f(y_1) dy_1. \end{aligned}$$

Using binomial expansion and simplifying, we get

$$\begin{aligned} F_{(\frac{m+1}{2}, s:m); l, k}(y_2) &= \frac{[F_Y(y_2)]^l}{B(k-l, l+1)} \int_{-\infty}^{\infty} \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t [\alpha_2 - \alpha_3 F_Y(y_1)]^t \\ &\quad \times [F_Y(y_1)]^{k-l-1} [1 - F_Y(y_1)]^l f_Y(y_1) dy_1 \\ &= \frac{[F_Y(y_2)]^l}{B(k-l, l+1)} \int_{-\infty}^{\infty} \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t \sum_{u=0}^t (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \\ &\quad \times [F_Y(y_1)]^{k-l+u-1} [1 - F_Y(y_1)]^l f_Y(y_1) dy_1 \\ &= \frac{[F_Y(y_2)]^l}{B(k-l, l+1)} \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t C(t) \end{aligned} \quad (17)$$

where

$$C(t) = \sum_{u=0}^t (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u B(k-l+u, l+1). \quad (18)$$

It follows from (17) that the probability density function of  $V_{(\frac{m+1}{2}, s; m); l, k}$  is

$$\begin{aligned} f_{(\frac{m+1}{2}, s; m); l, k}(y_2) &= \frac{[F_Y(y_2)]^{l-1}}{B(k-l, l+1)} f_Y(y_2) \\ &\times \left\{ l + \sum_{t=1}^l \binom{l}{t} [1 - F_Y(y_2)]^{t-1} [l - (l+t)F_Y(y_2)] C(t) \right\} \\ &= f_{l:l}(y_2) + \frac{1}{B(k-l, l+1)} \sum_{t=1}^l \frac{\binom{l}{t}}{\binom{l+t-1}{l}} [f_{l:l+t-1}(y_2) - f_{l+1:l+t}(y_2)] C(t) \end{aligned} \quad (19)$$

where  $f_{r:j}(y_2)$  is the probability density function of  $Y_{r:j}$ . The moments of  $V_{(\frac{m+1}{2}, s; m); l, k}$  are derived from (19) as:

$$\begin{aligned} \mu_{(\frac{m+1}{2}, s; m); l, k}^{(h)} &= \mathbb{E} \left[ V_{(\frac{m+1}{2}, s; m); l, k}^h \right] \\ &= \mu_{l:l}^{(h)} + \frac{1}{B(k-l, l+1)} \sum_{t=1}^l \frac{\binom{l}{t}}{\binom{l+t-1}{l}} \left( \mu_{l:l+t-1}^{(h)} - \mu_{l+1:l+t}^{(h)} \right) C(t) \end{aligned} \quad (20)$$

where

$$\mu_{r:j}^{(h)} = \mathbb{E} \left[ Y_{r:j}^h \right], \quad h = 1, 2, \dots; \quad l = 1, 2, \dots, k. \quad (21)$$

Applying the reduction formula for the gamma function, the distribution function in equation (17) simplifies to

$$\begin{aligned} F_{(\frac{m+1}{2}, s; m); l, k}(y_2) &= [F_Y(y_2)]^l \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t \sum_{u=0}^t (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \\ &\times \frac{(k-l)(k-l+1) \cdots (k-l+u-1)}{(k+1)(k+2) \cdots (k+u)} \\ &= [F_Y(y_2)]^l \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t \sum_{u=0}^t (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \\ &\times \frac{\prod_{j=0}^{u-1} \left( 1 - \frac{l-j}{k} \right)}{\prod_{j=1}^u \left( 1 + \frac{j}{k} \right)}. \end{aligned} \quad (22)$$

Keeping  $l$  fixed, as  $k \rightarrow \infty$ , the asymptotic distribution of  $V_{(\frac{m+1}{2}, s:m); l, k}$  converges to a limiting distribution characterized by the following distribution function:

$$\begin{aligned} F_{(\frac{m+1}{2}, s:m); l}(y_2) &= [F_Y(y_2)]^l \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t \sum_{u=0}^t (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \\ &= [F_Y(y_2)]^l \sum_{t=0}^l \binom{l}{t} [1 - F_Y(y_2)]^t (\alpha_2 - \alpha_3)^t \\ &= [F_Y(y_2)]^l \sum_{t=0}^l \binom{l}{t} [(\alpha_2 - \alpha_3)(1 - F_Y(y_2))]^t \\ &= [F_Y(y_2)]^l [1 + (\alpha_2 - \alpha_3)(1 - F_Y(y_2))]^l. \end{aligned} \quad (23)$$

(23) is the same as the distribution function of the maximum of  $l$  i.i.d. random variables, each with distribution function

$$F_Y(y_2) [1 + (\alpha_2 - \alpha_3)(1 - F_Y(y_2))].$$

The density function corresponding to (23) becomes

$$\begin{aligned} f_{(\frac{m+1}{2}, s:m); l}(y_2) &= l[F_Y(y_2)]^{l-1} f_Y(y_2) [1 + (\alpha_2 - \alpha_3)(1 - F_Y(y_2))]^{l-1} \\ &\quad \times [1 + (\alpha_2 - \alpha_3)(1 - 2F_Y(y_2))]. \end{aligned} \quad (24)$$

(24) may be represented as

$$\begin{aligned} f_{(\frac{m+1}{2}, s:m); l}(y_2) &= l[F_Y(y_2)]^{l-1} f_Y(y_2) [1 + (\alpha_2 - \alpha_3)(1 - F_Y(y_2))]^l \\ &\quad - (\alpha_2 - \alpha_3)l[F_Y(y_2)]^l f_Y(y_2) [1 + (\alpha_2 - \alpha_3)(1 - F_Y(y_2))]^{l-1} \\ &= (l!)^2 \left[ \frac{(\alpha_2 - \alpha_3)^l}{(2l)!} f_{l:2l}(y_2) + \sum_{t=0}^{l-1} \frac{(\alpha_2 - \alpha_3)^t}{(l-t)!(l+t)!} \right. \\ &\quad \left. \times \left\{ f_{l:l+t}(y_2) - \frac{(\alpha_2 - \alpha_3)(l-t)}{l+t+1} f_{l+1:l+t+1}(y_2) \right\} \right]. \end{aligned} \quad (25)$$

Moments of (25) are calculated as

$$\begin{aligned} \mu_{(\frac{m+1}{2}, s:m); l}^{(h)} &= \mathbb{E} \left[ V_{(\frac{m+1}{2}, s:m); l}^h \right] \\ &= (l!)^2 \left[ \frac{(\alpha_2 - \alpha_3)^l}{(2l)!} \mu_{l:2l}^{(h)} + \sum_{t=0}^{l-1} \frac{(\alpha_2 - \alpha_3)^t}{(l-t)!(l+t)!} \right. \\ &\quad \left. \times \left\{ \mu_{l:l+t}^{(h)} - \frac{(\alpha_2 - \alpha_3)(l-t)}{l+t+1} \mu_{l+1:l+t+1}^{(h)} \right\} \right] \end{aligned} \quad (26)$$

where

$$\mu_{r:j}^{(h)} = \mathbb{E} \left[ Y_{r:j}^h \right], \quad h = 1, 2, \dots$$

### 3. Applications

#### 3.1. Application to the Selection Problem

This section focuses on applying the findings from the preceding section to the Morgenstern-type bivariate uniform distribution. The distribution function corresponding to the bivariate uniform distribution is expressed as follows:

$$F_{X,Y}(x, y) = \frac{x}{\theta_1} \frac{y}{\theta_2} \left[ 1 + \alpha \left( 1 - \frac{x}{\theta_1} \right) \left( 1 - \frac{y}{\theta_2} \right) \right], \quad 0 < x < \theta_1, 0 < y < \theta_2, \\ -1 \leq \alpha \leq 1. \quad (27)$$

The distribution function of  $V_{(\frac{m+1}{2}, s:m); l, k}$  follows from (17) as,

$$F_{(\frac{m+1}{2}, s:m); l, k}(y_2) = \frac{\left(\frac{y_2}{\theta_2}\right)^l}{B(k-l, l+1)} \sum_{t=0}^l \binom{l}{t} \left(1 - \frac{y_2}{\theta_2}\right)^t C(t). \quad (28)$$

The density function of  $V_{(\frac{m+1}{2}, s:m); l, k}$  follows from (19) as

$$f_{(\frac{m+1}{2}, s:m); l, k}(y_2) = \left(\frac{y_2}{\theta_2}\right)^{l-1} \frac{1}{\theta_2} \left[ l + \sum_{t=1}^l \binom{l}{t} \left(1 - \frac{y_2}{\theta_2}\right)^{t-1} l - (l+t) \frac{y_2}{\theta_2} \right] \\ \times \frac{1}{B(k-l, l+1)} C(t), \quad 0 < y_2 < \theta_2. \quad (29)$$

Using  $\mu_{r:j} = \frac{r}{j+1} \theta_2$  in (20), we obtain

$$\mathbb{E} \left[ V_{(\frac{m+1}{2}, s:m); l, k} \right] = \mu_{l:l} + \frac{1}{B(k-l, l+1)} \sum_{t=1}^l \frac{\binom{l}{t}}{\binom{l+t-1}{l}} [\mu_{l:l+t-1} - \mu_{l+1:l+t}] C(t) \theta_2 \\ = \left[ \frac{l}{l+1} - \sum_{t=1}^l \sum_{u=0}^t \binom{l}{t} (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \frac{(k-l)_u t!}{(l+1)_{t+1} (k+1)_u} \right] \theta_2 \quad (30)$$

where

$$(n)_x = n(n+1)(n+2) \cdots (n+x-1).$$

Since

$$\mathbb{E} \left[ V_{\left(\frac{m+1}{2}, s:m\right); k, k} \right] = \mathbb{E} [Y_{k:k}] = \frac{k}{k+1} \theta_2,$$

we get

$$e_{\left(\frac{m+1}{2}, s:m\right); l, k} = \frac{k+1}{k} \left[ \frac{l}{l+1} - \sum_{t=1}^l \sum_{u=0}^t \binom{l}{t} (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \times \frac{(k-l)_u t!}{(l+1)_{t+1} (k+1)_u} \right]. \quad (31)$$

The asymptotic distribution of  $V_{\left(\frac{m+1}{2}, s:m\right); l, k}$  is given by (23) as

$$F_{\left(\frac{m+1}{2}, s:m\right); l}(y_2) = \left( \frac{y_2}{\theta_2} \right)^l \left[ 1 + (\alpha_2 - \alpha_3) \left( 1 - \frac{y_2}{\theta_2} \right) \right]^l, \quad 0 < y_2 < \theta_2 \quad (32)$$

and density function is given by

$$f_{\left(\frac{m+1}{2}, s:m\right); l}(y_2) = \frac{l}{\theta_2^l} y_2^{l-1} \left[ 1 + (\alpha_2 - \alpha_3) \left( 1 - \frac{y_2}{\theta_2} \right) \right]^{l-1} \times \left[ 1 + (\alpha_2 - \alpha_3) \left( 1 - \frac{2y_2}{\theta_2} \right) \right], \quad 0 < y_2 < \theta_2. \quad (33)$$

Also

$$\mathbb{E} \left[ V_{\left(\frac{m+1}{2}, s:m\right); l} \right] = (l!)^2 \left[ \frac{(\alpha_2 - \alpha_3)^l l}{(2l+1)!} + \sum_{t=0}^{l-1} \frac{(\alpha_2 - \alpha_3)^t}{(l+t+1)!(l-t)!} \left( l - (\alpha_2 - \alpha_3) \frac{(l-t)(l+1)}{l+t+2} \right) \right] \theta_2 \quad (34)$$

and

$$e_{\left(\frac{m+1}{2}, s:m\right); l} = (l!)^2 \left[ \frac{(\alpha_2 - \alpha_3)^l l}{(2l+1)!} + \sum_{t=0}^{l-1} \frac{(\alpha_2 - \alpha_3)^t}{(l+t+1)!(l-t)!} \left( l - (\alpha_2 - \alpha_3) \frac{(l-t)(l+1)}{l+t+2} \right) \right]. \quad (35)$$

It may be noted from (31) that  $e_{\left(\frac{m+1}{2}, s:m\right); l, k}$  is independent of the parameter  $\theta_2$ . For maximum effectiveness in the selection procedure, setting  $s = m$  is a reasonable choice. The values of  $e_{\left(\frac{m+1}{2}, m:m\right); l, k}$  for  $\alpha = 0.2$  (0.2) 1;  $m = 11$ (20) 51 and  $k = 20$  (10) 40 are tabulated in Table 1 using Wolfram Alpha for specified sets of values of  $l$ . Our goal is to find the number of measurements needed for the expensive marginal

variable to achieve  $e_{(\frac{m+1}{2}, m:m);l,k} \geq 0.95$ . Therefore, we limit the values of  $l$  to those that guarantee this specific efficiency threshold. It is observed from this table that  $e_{(\frac{m+1}{2}, m:m);l,k}$  increases with  $l$ ,  $m$  and  $\alpha$ , tends to 1 as  $l$  approaches to  $k$ . This selection method significantly minimizes the need for costly measurements. Table 2 demonstrates the number expensive measurements required to achieve the effectiveness measure  $e_{(\frac{m+1}{2}, m:m);l,k} \geq 0.95$  for all  $\alpha > 0$ . Table 3, prepared using Wolfram Alpha, presents the values of the effectiveness measure of selection  $e_{(\frac{m+1}{2}, m:m);l}$  in the asymptotic case. According to Table 3, only 17 highly expensive observations are sufficient to attain a selection effectiveness of 0.95 for all values of  $\alpha > 0$ , eliminating the need for a large sample size. This finding suggests that in environmental pollutant monitoring, collecting just 17 extreme values (which can be costly) is enough to achieve a specific efficiency or accuracy threshold  $e_{(\frac{m+1}{2}, m:m);l} \geq 0.95$ .

The gain due to additional information in the finite case is measured by

$$e_{(\frac{m+1}{2}, m:m);l+1,k} - e_{(\frac{m+1}{2}, m:m);l,k}$$

It is observed from the calculated values in Table 1 that the gain due to additional information is negligible for all  $l \geq 9$ ,  $m \geq 11$ ,  $k \geq 20$ , and  $\alpha \geq 0$ . Hence, we conclude that  $l \geq 10$  will yield an efficiency greater than or equal to 0.95 in the finite case. A comparison of  $e_{(\frac{m+1}{2}, m:m);l,k}$  and  $e_{(\frac{m+1}{2}, m:m);l}$  offers a reliable indication of the sample size at which the asymptotic results become valid. If  $m \geq 11$  and  $k \geq 1000$ , the asymptotic values closely agree with the finite values for all  $\alpha \geq 0$ .

The proposed method is statistically and economically efficient for identifying the best sampling unit among a large number of units requiring expensive measurements. Its applications span various fields, including environmental sampling, forestry, mining, ocean exploration, space science, petroleum resource evaluation, and medical research. This method is especially useful in situations where no prior information is available regarding the best sampling unit.

Table 1.: Values of  $e_{(\frac{m+1}{2}, m:m);l,k}$

$m$	$k$	$l$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
11	20	1	0.5542	0.5837	0.6132	0.6430	0.6729
11	20	2	0.7283	0.7547	0.7793	0.8020	0.8228
11	20	3	0.8124	0.8347	0.8545	0.8720	0.8875
11	20	4	0.8618	0.8807	0.8971	0.9112	0.9235
11	20	5	0.8942	0.9106	0.9245	0.9363	0.9465
11	20	6	0.9172	0.9315	0.9435	0.9537	0.9624
11	20	7	0.9342	0.9470	0.9576	0.9665	0.9740
11	20	8	0.9474	0.9589	0.9684	0.9763	0.9829
11	20	9	0.9579	0.9683	0.9769	0.9840	0.9899
11	20	10	0.9664	0.9759	0.9838	0.9902	0.9956
11	20	11	0.9735	0.9823	0.9895	0.9954	1
11	20	20	1	1	1	1	1
11	30	1	0.5455	0.5744	0.6035	0.6328	0.6623
11	30	2	0.7167	0.7427	0.7670	0.7893	0.8098
11	30	3	0.7995	0.8214	0.8409	0.8582	0.8735

*Continued on next page*

$m$	$k$	$l$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
11	30	4	0.8481	0.8667	0.8829	0.8969	0.9089
11	30	5	0.8800	0.8961	0.9098	0.9215	0.9315
11	30	6	0.9026	0.9167	0.9286	0.9386	0.9471
11	30	7	0.9194	0.9320	0.9424	0.9512	0.9586
11	30	8	0.9324	0.9437	0.9530	0.9608	0.9674
11	30	9	0.9427	0.9530	0.9614	0.9684	0.9743
11	30	10	0.9511	0.9605	0.9682	0.9746	0.9799
11	30	11	0.9580	0.9667	0.9738	0.9796	0.9845
11	30	30	1	1	1	1	1
11	40	1	0.5411	0.5698	0.5987	0.6277	0.6569
11	40	2	0.7109	0.7368	0.7608	0.7830	0.8033
11	40	3	0.7930	0.8148	0.8342	0.8513	0.8664
11	40	4	0.8413	0.8598	0.8758	0.8896	0.9016
11	40	5	0.8729	0.8889	0.9025	0.9141	0.9240
11	40	6	0.8953	0.9094	0.9211	0.9311	0.9395
11	40	7	0.9120	0.9245	0.9349	0.9436	0.9509
11	40	8	0.9248	0.9361	0.9454	0.9531	0.9596
11	40	9	0.9351	0.9453	0.9537	0.9606	0.9664
11	40	10	0.9434	0.9528	0.9604	0.9668	0.9720
11	40	11	0.9503	0.9590	0.9660	0.9718	0.9766
11	40	40	1	1	1	1	1
31	20	1	0.5578	0.5907	0.6235	0.6564	0.6894
31	20	2	0.7316	0.7607	0.7874	0.8117	0.8335
31	20	3	0.8152	0.8396	0.8609	0.8793	0.8953
31	20	4	0.8642	0.8848	0.9023	0.9171	0.9297
31	20	5	0.8964	0.9141	0.9289	0.9412	0.9515
31	20	6	0.9191	0.9346	0.9474	0.9579	0.9667
31	20	7	0.9359	0.9497	0.9610	0.9702	0.9778
31	20	8	0.9489	0.9614	0.9714	0.9795	0.9863
31	20	9	0.9592	0.9706	0.9796	0.9869	0.9930
31	20	10	0.9677	0.9780	0.9862	0.9929	0.9984
31	20	11	0.9747	0.9842	0.9917	0.9978	1
31	20	20	1	1	1	1	1
31	30	1	0.5490	0.5813	0.6137	0.6460	0.6784
31	30	2	0.7200	0.7487	0.7750	0.7988	0.8203
31	30	3	0.8023	0.8263	0.8472	0.8654	0.8811
31	30	4	0.8505	0.8708	0.8880	0.9026	0.9149
31	30	5	0.8822	0.8996	0.9141	0.9263	0.9364
31	30	6	0.9045	0.9198	0.9323	0.9427	0.9514
31	30	7	0.9211	0.9347	0.9457	0.9548	0.9623
31	30	8	0.9339	0.9461	0.9560	0.9640	0.9706
31	30	9	0.9441	0.9552	0.9640	0.9713	0.9772
31	30	10	0.9524	0.9625	0.9706	0.9772	0.9825
31	30	11	0.9592	0.9686	0.9760	0.9820	0.9869
31	30	30	1	1	1	1	1
31	40	1	0.5445	0.5766	0.6087	0.6408	0.6730
31	40	2	0.7142	0.7426	0.7687	0.7924	0.8136
31	40	3	0.7958	0.8196	0.8404	0.8584	0.8740

*Continued on next page*

$m$	$k$	$l$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
31	40	4	0.8437	0.8638	0.8808	0.8953	0.9076
31	40	5	0.8750	0.8924	0.9068	0.9188	0.9289
31	40	6	0.8972	0.9124	0.9248	0.9351	0.9436
31	40	7	0.9136	0.9271	0.9381	0.9470	0.9545
31	40	8	0.9263	0.9385	0.9482	0.9562	0.9628
31	40	9	0.9364	0.9475	0.9563	0.9635	0.9694
31	40	10	0.9446	0.9548	0.9628	0.9693	0.9747
31	40	11	0.9515	0.9608	0.9682	0.9741	0.9790
31	40	40	1	1	1	1	1
51	20	1	0.5587	0.5923	0.6260	0.6597	0.6934
51	20	2	0.7324	0.7621	0.7893	0.8140	0.8360
51	20	3	0.8159	0.8407	0.8624	0.8810	0.8971
51	20	4	0.8648	0.8858	0.9035	0.9185	0.9311
51	20	5	0.8969	0.9149	0.9299	0.9423	0.9527
51	20	6	0.9195	0.9353	0.9482	0.9589	0.9677
51	20	7	0.9363	0.9504	0.9617	0.9710	0.9786
51	20	8	0.9493	0.9619	0.9720	0.9803	0.9870
51	20	9	0.9596	0.9711	0.9802	0.9876	0.9936
51	20	10	0.9680	0.9785	0.9868	0.9935	0.9990
51	20	11	0.9749	0.9846	0.9923	0.9984	1
51	20	20	1	1	1	1	1
51	30	1	0.5498	0.5829	0.6161	0.6492	0.6824
51	30	2	0.7207	0.7500	0.7768	0.8010	0.8227
51	30	3	0.8030	0.8274	0.8487	0.8671	0.8829
51	30	4	0.8511	0.8717	0.8892	0.9039	0.9163
51	30	5	0.8826	0.9004	0.9151	0.9274	0.9376
51	30	6	0.9049	0.9205	0.9332	0.9436	0.9523
51	30	7	0.9214	0.9353	0.9464	0.9556	0.9631
51	30	8	0.9342	0.9466	0.9566	0.9647	0.9714
51	30	9	0.9444	0.9557	0.9647	0.9719	0.9779
51	30	10	0.9526	0.9630	0.9712	0.9778	0.9832
51	30	11	0.9595	0.9690	0.9765	0.9825	0.9875
51	30	30	1	1	1	1	1
51	40	1	0.5454	0.5782	0.6111	0.6440	0.6769
51	40	2	0.7149	0.7440	0.7706	0.7946	0.8161
51	40	3	0.7964	0.8207	0.8418	0.8601	0.8758
51	40	4	0.8442	0.8647	0.8820	0.8966	0.9089
51	40	5	0.8755	0.8932	0.9078	0.9199	0.9300
51	40	6	0.8976	0.9130	0.9257	0.9360	0.9446
51	40	7	0.9140	0.9277	0.9388	0.9479	0.9554
51	40	8	0.9267	0.9390	0.9489	0.9569	0.9635
51	40	9	0.9368	0.9479	0.9569	0.9641	0.9700
51	40	10	0.9450	0.9552	0.9633	0.9699	0.9752
51	40	11	0.9518	0.9612	0.9687	0.9747	0.9796
51	40	40	1	1	1	1	1

Table 2.: Values of  $l$  for which  $e_{(\frac{m+1}{2},m:m);l,k} \geq 0.95$ 

$m$	$k$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
11	10	6	5	5	4	4
11	20	9	8	7	6	6
11	1000	17	15	13	12	11
31	20	9	8	7	6	5
31	30	10	9	8	7	6
31	40	11	10	9	8	7
31	1000	16	14	13	11	10
51	20	9	8	7	6	5
51	30	10	9	8	7	6
51	40	11	10	9	8	7
51	1000	16	14	12	11	10
101	50	13	11	10	9	8
101	100	14	12	11	9	8
101	1000	16	14	12	11	10

Table 3.: Values of  $e_{(\frac{m+1}{2},m:m);l}$ 

$m$	$l$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
11	1	0.5279	0.5559	0.5841	0.6125	0.6410
11	17	0.9518	0.9575	0.9622	0.9659	0.9691
11	21	0.9606	0.9654	0.9692	0.9723	0.9748
11	41	0.9794	0.9820	0.9840	0.9856	0.9869
11	61	0.9861	0.9878	0.9892	0.9902	0.9911
11	81	0.9895	0.9908	0.9919	0.9927	0.9933
11	101	0.9916	0.9926	0.99345	0.9941	0.9947
51	1	0.5321	0.5641	0.5962	0.6283	0.6604
51	11	0.9286	0.9378	0.9451	0.9510	0.9557
51	17	0.9527	0.9590	0.9638	0.9677	0.9709
51	21	0.9614	0.9666	0.9705	0.9738	0.9763
51	41	0.9799	0.9826	0.9847	0.9864	0.9878
51	61	0.9864	0.9883	0.9897	0.9908	0.9917
51	81	0.9897	0.9912	0.9922	0.9931	0.9937
51	101	0.9918	0.9929	0.9938	0.9944	0.9950

### 3.2. Application to Inference

This section focuses on estimating the scale parameter  $\theta_2$  corresponding to the expensive marginal variable  $Y$ , utilizing selected second order concomitants derived from the model specified in equation (27). The fact that only the marginal information on  $Y$  is available, prompted us to infer that UMVUE of  $\theta_2$  will be a function of the sample maximum. It is observed that the expected value of the sample maximum based on  $l$  expensive measurements is approximately equal to  $E \left[ V_{(\frac{m+1}{2},m:m);l,k} \right]$ . An estimator based on  $V_{(\frac{m+1}{2},m:m);l,k}$  is expected to exhibit greater efficiency and reduced cost compared to the UMVUE of  $\theta_2$ . Accordingly, we introduce the following estimator for

$\theta_2$

$$\hat{\theta}_{2[l,k]} = \frac{V_{(\frac{m+1}{2}, m:m);l,k}}{a_{[l,k]}} \quad (36)$$

where

$$a_{[l,k]} = \frac{l}{l+1} - \sum_{t=1}^l \sum_{u=0}^t \binom{l}{t} (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \frac{(k-l)_u t!}{(l+1)_{t+1} (k+1)_u}. \quad (37)$$

From (20)

$$\begin{aligned} E \left[ V_{(\frac{m+1}{2}, m:m);l,k}^2 \right] &= \left[ \frac{l}{l+2} - 2 \sum_{t=1}^l \sum_{u=0}^t \binom{l}{t} (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \right. \\ &\quad \left. \frac{(k-l)_u t!}{(l+2)_{t+1} (k+1)_u} \right] \theta_2^2 \\ &= b_{[l,k]} \theta_2^2 \end{aligned} \quad (38)$$

where

$$b_{[l,k]} = \frac{l}{l+2} - 2 \sum_{t=1}^l \sum_{u=0}^t \binom{l}{t} (-1)^u \binom{t}{u} \alpha_2^{t-u} \alpha_3^u \frac{(k-l)_u t!}{(l+2)_{t+1} (k+1)_u}. \quad (39)$$

The variance of the estimator  $\hat{\theta}_{2[l,m]}$  is

$$V \left( \hat{\theta}_{2[l,k]} \right) = \theta_2^2 \left[ \frac{b_{[l,k]}}{a_{[l,k]}^2} - 1 \right]. \quad (40)$$

It is worthwhile to compare  $\hat{\theta}_{2[l,m]}$  with the UMVUE of  $\theta_2$ . The UMVUE of  $\theta_2$  is

$$\hat{\theta}_2 = \frac{l+1}{l} Y_{l:l} \quad (41)$$

and

$$V(\hat{\theta}_2) = \frac{1}{l(l+2)} \theta_2^2. \quad (42)$$

The relative efficiency of  $\hat{\theta}_{2[l,k]}$  over  $\hat{\theta}_2$  is

$$R_{[l,k]} = \frac{a_{[l,k]}^2}{l(l+2) [b_{[l,k]} - a_{[l,k]}^2]}. \quad (43)$$

It may be noted that, the proposed estimator is a function of  $\alpha$ . Scaria and Nair (2008) proposed an estimate of  $\alpha$  as

$$\hat{\alpha} = 3 \left[ \frac{Y_{(\frac{m+1}{2}, m:m); [l:l]} - Y_{(\frac{m+1}{2}, m:m); [1:l]}}{Y_{(\frac{m+1}{2}, m:m); l:l} - Y_{(\frac{m+1}{2}, m:m); 1:l}} \right] \quad (44)$$

if the estimated value lies in  $(-1, 1)$  and  $1(-1)$  if the value is  $\geq 1$  ( $\leq -1$ ).  $\hat{\alpha}$  is based only on the selected expensive measurements. The maximum likelihood estimator of  $\alpha$  can be obtained using suitable softwares.

The computed values of  $R_{[l:k]} \alpha = 0.2$  (0.2)1,  $m = 11$  (20) 51,  $k = 20$  (10) 40 and,  $l = 5$  (5) 15 are tabulated in Table 4 using Wolfram Alpha. The table reveals that the relative efficiency  $R_{[l:k]}$  increases with  $\alpha, m$  and  $l$ . Also,  $\hat{\theta}_2[l, k]$  is significantly more efficient than the UMVUE (Uniformly Minimum Variance Unbiased Estimator)  $\hat{\theta}_2$ . Notably, the efficiency of the proposed estimator increases rapidly when X and Y are moderately correlated.

Table 4.: Values of  $R_{[l:k]}$

$l$	$m$	$k$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
5	11	20	1.2406	1.5398	1.9063	2.3478	2.8705
5	11	30	1.2407	1.5402	1.9073	2.3500	2.8744
5	11	40	1.2407	1.5404	1.9078	2.3511	2.8764
5	31	20	1.2739	1.6214	2.0533	2.5786	3.2034
5	31	30	1.2740	1.6215	2.0535	2.5790	3.2040
5	31	40	1.2740	1.6215	2.0536	2.5792	3.2043
5	51	20	1.2818	1.6411	2.0895	2.6363	3.2875
5	51	30	1.2818	1.6411	2.0895	2.6364	3.2878
5	51	40	1.2818	1.6411	2.0896	2.6365	3.2879
10	11	20	1.2874	1.6358	2.0489	2.5296	3.0797
10	11	30	1.2876	1.6366	2.0510	2.5339	3.0872
10	11	40	1.2877	1.6370	2.0521	2.5360	3.0911
10	31	20	1.3270	1.7303	2.2142	2.7817	3.4341
10	31	30	1.3270	1.7304	2.2146	2.7824	3.4353
10	31	40	1.3271	1.7305	2.2148	2.7827	3.4359
10	51	20	1.3363	1.7529	2.2543	2.8434	3.5217
10	51	30	1.3363	1.7529	2.2544	2.8437	3.5221
10	51	40	1.3364	1.7530	2.2545	2.8438	3.5224
15	11	20	1.3080	1.6762	2.1066	2.6005	3.1587
15	11	30	1.3083	1.6774	2.1097	2.6068	3.1698
15	11	40	1.3084	1.6781	2.1113	2.6101	3.1755
15	31	20	1.3504	1.7766	2.2808	2.8642	3.5277
15	31	30	1.3505	1.7768	2.2812	2.8652	3.5295
15	31	40	1.3505	1.7769	2.2814	2.8658	3.5304
15	51	20	1.3604	1.8004	2.3225	2.9280	3.6174
15	51	30	1.3604	1.8005	2.3227	2.9283	3.6181
15	51	40	1.3604	1.8005	2.3228	2.9285	3.6185

## Acknowledgement

The authors gratefully acknowledge the referee's valuable suggestions that helped refine the earlier version of this manuscript.

## References

- [1] Ahsanullah, M. (2009). Records and concomitants. *Bulletin of the Malaysian Mathematical Sciences Society*. 32(2), 101-117.
- [2] Bairamov, I., Kotz, S. and Bekci, M. (2001). New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics. *Journal of Applied Statistics*. 28 (5), 521-536.
- [3] Barakat, H. M., Alawady, M. A., Taher, T. S. and Husseiny, I. A. (2025). Second order concomitants of order statistics from bivariate Cambanis family: some information measures—estimation. *Communications in Statistics - Simulation and Computation*. 54(8), 3195-3221. <https://doi.org/10.1080/03610918.2024.2344023>
- [4] Bhattacharya, P.K. (1974). Convergence of sample paths of normalized sums of induced order statistics. *The Annals of Statistics*. 2, 1034-1039.
- [5] Beg, M. I. and Ahsanullah, M. (2008). Concomitants of generalized order statistics from Farlie-Gumbel-Morgenstern distributions. *Statistical Methodology*. 5, 1-20.
- [6] David, H.A. (1973). Concomitants of order statistics. *Bulletin of International Statistical Institute*. 45, 295-300.
- [7] David, H.A. and Galambos, J. (1974). The asymptotic theory of concomitants of order statistics. *Journal of Applied Probability*. 11, 762-770.
- [8] David, H.A. and Nagaraja, H. N. (1998). Concomitants of order statistics. In: Balakrishnan, N. and Rao, C.R. (eds.), *Hand book of Statistics*, Elsevier Science, New York, 16, 487- 513.
- [9] Eryilmaz, S. (2005). Concomitants in a sequence of independent non identically distributed random vectors. *Communications in Statistics-Theory and Methods*. 34, 1925-1933.
- [10] Feinberg, F.M (1991). Modelling optimal cutoff severity: a comparison of discrete and continuous cut off rules under imperfect information, working paper, Fuqua School of Business
- [11] Feinberg, F.M and J. Huber (1996). A theory of cut off formation under imperfect information, *Management Science*, 42(1), 65-84.
- [12] Koshti, R.D. and Kamalja, K.K. (2024). A review on concomitants of order statistics and its application in parameter estimation under ranked set sampling. *Journal Korean Statistical Society*, 53, 65–99. <https://doi.org/10.1007/s42952-023-00235-2>.
- [13] Morgenstern, D. (1956). Einfache Beispiele zweidimensionaler Verteilungen, *Mitteilungsblatt für Mathematische Statistik*, 8, 234-235.
- [14] Nagaraja, H. N. and David, H.A. (1994). Distribution of the maximum of concomitants of selected order statistics, *The Annals of Statistics*, 22(1), 478-494.
- [15] Scaria, J. (2003). Concomitants of Order Statistics from Morgenstern Family. Ph.D thesis, Department of Statistics, Cochin University of Science and Technology, Cochin.
- [16] Scaria, J. and Mohan, S. (2021). Dependence concepts and reliability application of concomitants of order statistics from the Morgenstern family, *Journal of Statistical Theory and Applications*. 20(2), 193–203.
- [17] Scaria, J. and Nair, N.U. (1999). On concomitants of order statistics from Morgenstern family. *Biometrical Journal*. 41(4), 483-489.
- [18] Scaria, J. and Nair, N.U. (2005). Distribution of the maximum of concomitants of selected order statistics from the Morgenstern family of distributions. *Journal of Applied Statistical Science*. 14(5), 251-262.

- [19] Scaria, J. and Nair, N.U. (2008). Distribution of extremes of  $r$ th concomitant from the Morgenstern family. *Statistical Papers*. 49(1), 109-119.
- [20] Scaria, J. and Thomas, B. (2008). Concomitants of order statistics from Generalized Morgenstern family, *STARS: International Journal*, 2(2), 153-162.
- [21] Scaria, J. and Thomas, B. (2013). Second order concomitants from the Morgenstern family of distributions, *Journal of Applied Statistical Science*, 21(1), 63-76.
- [22] Thomas, B. (2015). Concomitants of Order Statistics from Bivariate Generalized Morgenstern and Cambanis Family. Ph.D thesis, Department of Statistics, Nirmala College, Muvattupuzha.
- [23] Thomas, B. and Scaria, J. (2011). Concomitants of order statistics from bivariate Cambanis family, *Science & Society*, 9(1), 49-62.
- [24] Yang, S.S. (1977). General distribution theory of the concomitants of order statistics. *The Annals of Statistics*. 5, 996-1002.
- [25] Yeo, W.B. and David, H.A. (1984). Selection through an associated characteristic with applications to the random effects model. *Journal of the American Statistical Association*. 79, 399-405.